Semestral

Elementary Number Theory

Instructor: Ramdin Mawia   Marks: **50**   **Time:** November 24, 2023; 14:00–17:00.

Attempt FIVE problems, selecting at least one from each section. Each question carries 10 marks. The maximum you can score is 50.

Counting, Primes, Divisibility, Congruences

1. State whether the following statements are true or false, with complete justifications: **10**
   i. If $p > 7$ is a prime, then $1 + 2(p-3)!$ is not a power of p.
   ii. If 101 integers are chosen from the set $\{1, 2, \cdots, 200\}$, then among the integers chosen, there are two such that one of them divides the other.
2. Find the last 2023 digits of $2023^{10^{2022}}$ in the usual decimal notation. Justify all the steps. **10**

Primitive roots and Quadratic residues

3. What do you mean by a *primitive root* modulo a positive integer n? Give, without proof, the complete list of positive integers n which have primitive roots. **10**
   i. Prove that 2 is a primitive root modulo 11.
   ii. In a complete residue system modulo 121, how many members are congruent to 2 modulo 11? Are all of them primitive roots modulo 121? If not, list the ones that are not primitive roots modulo 121. [*Hint.* Basic version of Hensel's lemma.]
4. State the law of Quadratic Reciprocity, and the complementary formulas for $(-1/p), (2/p)$. Use them to evaluate the Legendre symbols $(113/127)$, $(1999/2011)$, $(2027/2029)$, $(2039/2053)$. **10**

Arithmetic functions

5. Let $\sigma(n)$ denote, as usual, the sum of the positive divisors of $n \in \mathbb{Z}^+$. Prove the following: **10**
   i. σ is multiplicative but not completely multiplicative.
   ii. $\sigma(n) < n(1 + \log n)$ for all $n \geqslant 2$.
   iii. $\sum_{n \leqslant x} \sigma(n) = (\pi^2/12)x^2 + O\left(x \log x\right)$.
6. State the Möbius inversion formula. Use it to show that the following two formulas are equivalent:¶ **10**
   - $\tau(mn) = \sum_{d|(m,n)} \mu(d)\tau(m/d)\tau(n/d)$ for all positive integers m and n.
   - $\tau(m)\tau(n) = \sum_{d|(m,n)} \tau(mn/d^2)$ for all positive integers m and n.

   Here, τ is the usual divisor-counting function.

Binary quadratic forms, Sums of squares

7. i. Are there Pythagorean triplets which contain 2023 as a member? If yes, find all of them. **10**
   ii. Prove that, for any prime p, there are integers a and b such that p divides $a^2 + b^2 + 1$.
8. Show that $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ are the only reduced quadratic forms of discriminant -20. Prove that the first of these forms does not represent 2, but that the second one does. Deduce that the class number $H(-20) = 2$. Show that an odd prime p is represented by at least one of these forms if and only if $p = 5$ or $p \equiv 1, 3, 7, \text{ or } 9 \pmod{20}$. **10**

¶You are ***not*** asked to prove that the two formulas are correct (which they are)!